MATH 2613, Foundations of Mathematics
Practice Test 1, Fall 2009

Your test will be of similar content and material, but the questions will be different and
the questions may be asked if a different manner. The test questions will come from all
the material we have covered in class.

This is a closed book, closed note test, and should be your work only. You may use a
calculator during this exam. Stay calm, read all the instructions, show your work and
write your answers on this test, and ask the instructor for clarification if you are confused
as to what is being asked. Some unjustified answers will receive minimal credit.

1. Circle T if the symbolic sentence is true for the indicated universe and circle F if the
symbolic sentence is false for the indicated universe.

T  (F)  (\exists x)(\forall y)(x + y = 0) (Real Numbers)
T  (F)  (\exists x)(2x + 3 = 6x + 7) (Natural Numbers)  \quad 2x + 3 = 6x + 7
T  (F)  (\forall x)(x^2 + 6x + 5 \geq 0) (Real Numbers)  \quad \text{Vertex} \quad -\frac{-4}{2} = 4x \quad \text{(not a}
T  (F)  (\forall x)(\exists y)(x + y = 0) (Natural Numbers)  \quad x = -3
\quad y = 9 - 18 + 5 = -4
\quad \text{natural #)}

2a. Translate the sentence “There is an isosceles triangle that is a right triangle” into a
symbolic sentence with quantifiers. Use the universe of all triangles.

P(x) = x is an isosceles \Delta
Q(x) = x is a right \Delta

(\exists x)(P(x) \land Q(x))

b. For your symbolic sentence in 2a, write the denial as a symbolic sentence in as simplified
form as possible.

\neg (\exists x)(P(x) \land Q(x)) \equiv (\forall x)(\neg (P(x) \land Q(x))
\equiv (\forall x)(P(x) \Rightarrow \neg Q(x))

c. Translate the symbolic sentence in 2b into an English sentence. (There should be no
symbols in this sentence.)

All isosceles triangles are not right triangles.
3. Fill in the blanks for the following sentences:

a. $7 + 6 = 14$ if $5 + 5 = 10$.
   \[
   \text{Antecedent} = \underline{5 + 5 = 10} \quad \text{(T)}
   \]
   \[
   \text{Consequent} = \underline{7 + 6 = 14} \quad \text{(F)}
   \]
   True or False? \underline{False}

b. $6$ is prime is sufficient for $3 > 6$.
   \[
   \text{Antecedent} = \underline{6 \text{ is prime}} \quad \text{(F)}
   \]
   \[
   \text{Consequent} = \underline{3 > 6} \quad \text{(F)}
   \]
   True or False? \underline{True}

4. Make a truth table for the propositional form $P \land (Q \leftrightarrow R)$.

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$R$</th>
<th>$Q \leftrightarrow R$</th>
<th>$P \land (Q \leftrightarrow R)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>
5. Prove the following statement by contraposition:

If \( p^2 \) is even, then \( p \) is even.

Suppose \( p \) is odd.

By defn of odd, \( \exists n \) which is an integer, s.t. \( p = 2n + 1 \).

Squaring both sides, we have \( p^2 = (2n + 1)^2 \)

\[ = 4n^2 + 4n + 1 \] by distributive,

Commutative + assoc. Property

So \( p^2 = 2(2n^2 + 2n) + 1 \) by distributive + assoc. Properties

Let \( m = 2n^2 + 2n \) which is an integer since integers are closed under multiplication and addition.

Thus \( p^2 = 2m + 1 \) for an integer \( m \) by substitution.

Hence by defn \( p^2 \) is odd.

By the contrapositive, we know if \( p^2 \) is even, then \( p \) is odd.

6. Make a truth table for the statement \( (Q \lor \sim P) \iff (P \Rightarrow Q) \) making sure to show all relevant columns. Then determine if it is a tautology, a contradiction, or neither. Explain your answer.

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( \neg P )</th>
<th>( Q \lor \neg P )</th>
<th>( P \Rightarrow Q )</th>
<th>((Q \lor \neg P) \iff (P \Rightarrow Q))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
</tbody>
</table>

Tautology because we end up with all T's.
7. Use the following statement for this problem:

If $A$ and $B$ are nonsingular matrices, then the determinant of $AB$ is not 0.

a. Outline a direct proof of this theorem.

Suppose $A$ and $B$ are nonsingular matrices.

Then the determinant of $AB$ is not 0.

b. Outline a proof by contraposition of this theorem.

Suppose the determinant of $AB$ is 0.

Then either $A$ or $B$ is a singular matrix (not nonsingular).

Hence by the contrapositive, if $A$ and $B$ are nonsingular matrices, then the determinant of $AB$ is not 0.

c. Outline a proof by contradiction of this theorem.

Suppose $A$ and $B$ are nonsingular matrices.

Suppose the determinant of $AB$ is 0.

Find a contradiction.

So, we must have that the determinant of $AB$ is 0 is a false statement.

Thus the determinant of $AB$ is not 0.
8. Consider the statement "If squares have three sides, then triangles have four sides."

a. Write the converse of this statement.

If triangles have four sides, then squares have three sides.

b. Write the inverse of this statement.

If squares do not have three sides, then triangles do not have four sides.

c. Write the contrapositive of this statement.

If triangles do not have four sides, then squares do not have three sides.

9. Let $a$ and $b$ be real numbers. Use a direct proof to prove that $|ab| = |a||b|$.

Pf. Suppose $a$, $b$ are real numbers.

**Case I:** $a \geq 0$, $b \geq 0$.

Then $|ab| = ab$ by defn of absolute value.

$= |a||b|$ by defn of absolute value.

**Case II:** $a \geq 0$, $b < 0$.

Then $|ab| = -(ab)$ by defn of absolute value.

$= a(-b)$ by commutative & associativity.

$= |a||b|$ by defn of absolute value.

**Case III:** $a < 0$, $b \geq 0$.

Then $|ab| = -(ab)$ by defn of absolute value.

$=(-a)b$ by associativity.

$= |a||b|$ by defn of absolute value.

**Case IV:** $a < 0$, $b < 0$.

Then $|ab| = ab$ by defn of absolute value.

$= (-1)(-1)ab$ since $(-1)(-1) = 1$ which is multiplicative identity.

$= (-a)(-b)$ by commutativity and associativity.

$= |a||b|$ by defn of absolute value.

In all cases, $|ab| = |a||b|$.