MATH 3213, Abstract Algebra
Test 1, Fall 2009

This is a closed book, closed note test, and should be your work only. You may use a calculator during this exam. Stay calm, read all the instructions, show your work and write your answers on this test, and ask the instructor for clarification if you are confused as to what is being asked.

1. State the definition of a subgroup in its entirety.

Suppose \( G \) is a group. A nonempty subset \( H \) of \( G \) is a subgroup if \( H \) is a group using the operation of \( G \).

2. What is the center of \( D_4 \)?

\[
Z(D_4) = \{ R_0, R_{180} \}
\]

3. State the definition of the centralizer of an element \( a \) in a group \( G \).

\[
C(a) = \{ \chi \in G : \chi a = a \chi \}
\]

4a. Write down the Cayley Table for the set \( \{5, 15, 25, 35\} \) under multiplication modulo 40.

<table>
<thead>
<tr>
<th></th>
<th>5</th>
<th>15</th>
<th>25</th>
<th>35</th>
</tr>
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<tbody>
<tr>
<td>5</td>
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<td>5</td>
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</table>

b. What is the identity element for this group?

25

c. Find the orders of each element in the group.

<table>
<thead>
<tr>
<th>Element</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>2</td>
</tr>
<tr>
<td>25</td>
<td>1</td>
</tr>
<tr>
<td>35</td>
<td>2</td>
</tr>
</tbody>
</table>

\[
5 \cdot 5 = 25 \quad 8/125
\]

\[
15 \cdot 15 = 25
\]

Typeset by AMSTEX
5. Circle T if the sentence is true and circle F if the sentence is false. For all of these problems, assume that $G$ represents a group.

- **T** F If $G$ is an abelian group, then $Z(G) = G$.

- **T** F The set $\{0, 1, 2, 3\}$ is a group under the operation of multiplication mod 4.

$$
\begin{array}{c|cccc}
0 & 0 & 0 & 0 & 0 \\
1 & 1 & 2 & 3 & 1 \\
2 & 2 & 1 & 8 & 7 \\
3 & 3 & 4 & 5 & 6 \\
4 & 4 & 3 & 2 & 1 \\
5 & 5 & 6 & 7 & 8 \\
6 & 6 & 5 & 4 & 3 \\
7 & 7 & 8 & 1 & 2 \\
8 & 8 & 7 & 6 & 5 \\
\end{array}
$$

- **T** F If the order of a group $G$ is 8, then there must be an element of order 8 in the group.

There may be, but they don't have to be.

- **T** F In every group $G$, $\forall a, b \in G$, $(ab)^{-1} = a^{-1}b^{-1}$

Only if $G$ is abelian

6. Complete the partial Cayley group table given below:

$$
\begin{array}{c|cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
\hline
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
2 & 2 & 1 & 8 & 7 & 6 & 5 & 4 & 3 \\
3 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 \\
4 & 4 & 3 & 2 & 1 & 8 & 7 & 6 & 5 \\
5 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 \\
6 & 6 & 5 & 4 & 3 & 2 & 1 & 8 & 7 \\
7 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 \\
8 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\
\end{array}
$$

- $5 \cdot 6 = 5 \cdot (5 \cdot 2) = 1 \cdot 2$
- $5 \cdot 7 = 5 \cdot (5 \cdot 3) = 1 \cdot 3$
- $6 \cdot 5 = (5 \cdot 2) \cdot 5 = 5 \cdot (2 \cdot 5) = 5 \cdot 6 = 2$
- $6 \cdot 7 = (3 \cdot 4) \cdot 7 = 3 \cdot (4 \cdot 7) = 3 \cdot 6 = 8$
- $7 \cdot 5 = (5 \cdot 3) \cdot 5 = 5 \cdot (3 \cdot 5) = 5 \cdot 1 = 3$

$7 \cdot 6 = (5 \cdot 3) \cdot 6 = 5 \cdot (3 \cdot 6) = 5 \cdot 8 = 4$
7. Prove that in a group $G$, there is a unique identity element.

Suppose $G$ is a group.
Since $G$ is a group we know that $G$ has at least one identity element say $e$.
Suppose $G$ has two identity elements, that is suppose $e'$ is another identity element of $G$.

Since $e$ is an identity $ee' = e' = e'e$.
Since $e'$ is an identity $ee' = e = e'e$.
As $ee' = e'$ and $ee' = e$, by transitivity, $e = e'$.
Thus a group has a unique identity element.
8. Prove that the center of a group $G$ is a subgroup of $G$.

Suppose $G$ is a group.

Let $Z(G) = \{ x \in G : xa = ax \ \forall a \in G \}$

Since $ea = ae \ \forall a \in G$, $e \in Z(G)$ so $Z(G)$ is not empty.

Suppose $x, y \in Z(G)$.

By defn of center $xa = ax$ and $ay = ya \ \forall a \in G$.

$(x'y)a = x'(ya)$ by associativity

$= x'(ay)$ since $ay = ya$ (substitution)

$= (x'a)y$ by associativity

$= (a^{-1}x^{-1})y$ by socks-shoes

As $xa = ax$, we know $(xa)x^{-1} = (ax)x^{-1}$ by mul

$= (xa^{-1})y$ since $a^{-1} \in G$ and $x$ commutes with every element of $G$ by defn of center

$= (ax^{-1})y$ by socks-shoes

$= a(x'y)$ by associativity

Since $x'y$ commutes with $a$, $x'y \in Z(G)$.

So by the one step subgroup test, $Z(G)$ is a Subgroup.
9. Suppose that $G$ is a group with the following property: Whenever $a, b, c$ belong to $G$ with $ab = ca$, then $b = c$. Prove that $G$ is abelian.

Suppose $G$ is a group. 
Suppose whenever $a, b, c \in G$ with $ab = ca$, then $b = c$.

Suppose $x, y \in G$.

Let $b = xy \in G$ since $G$ is closed.

Let $c = yx \in G$ since $G$ is closed.

Let $a = y$.

Then $ab = y(xy)$ by substitution.

$= (yx)y$ by associativity.

$= ca$ by substitution.

Thus by hypothesis, we know $b = c$ or that $xy = yx$.

Thus $G$ is Abelian.