MATH 4473, Seminar - Discrete Mathematical Structures
Test 1, Spring 2010

This is a closed book, closed note test, and should be your work only. You may use a calculator during this exam. Stay calm, read all the instructions, show your work and write your answers on this test. Ask the instructor for clarification if you are confused as to what is being asked.

1. How many strings of eight lowercase English letters are there
a. that contain no vowels, if letters can be repeated? 

\[ 21^8 \approx 3.782285 \times 10^{10} \]

b. that start with a vowel, if letters cannot be repeated?

\[ 5 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 \approx 1.211364 \times 10^{10} \]

2. How many numbers must be selected from the set \{1, 4, 5, 8, 10, 13, 14, 17\} to guarantee that at least one pair of these numbers add up to 18?

\[ \{1, 17\}, \{4, 14\}, \{5, 13\}, \{8, 10\} \]

By Pigeonhole Principle, you need to select \[ 5 \] numbers.

3. How many ways are there to distribute three balls into eight boxes if \( A, B, C \)
a. the balls are labeled, but the boxes are unlabeled?

\[ \begin{array}{c}
\{A, B, C\} \\
\{A, B, C\} \text{ or } \{A, C\} \text{ or } \{B, C\} \\
\{A\} \text{ or } \{B\} \text{ or } \{C\}
\end{array} \]

b. the balls are unlabeled, but the boxes are labeled?

\[ \binom{10}{3} = \frac{10!}{9! \cdot 8} = \frac{120}{6} = 20 \]
4. How many ways can a set of two positive integers less than 100 be chosen?

Order doesn't matter

\[ C(99, 2) = \frac{99 \cdot 98}{2} = 4851 \]

5. What is the probability of the following events when we randomly select a permutation of the 26 lowercase letters of the English alphabet?

a. \( a \) is the first letter of the permutation and \( z \) is the last letter.

\[ \frac{24!}{26!} = \frac{1}{26 \cdot 25} \approx 0.0015 \]

b. \( a \) and \( z \) are next to each other in the permutation.

\[ \frac{25!}{26!} = \frac{2}{26} = \frac{1}{13} \approx 0.077 \]

6. Suppose that 8% of the patients tested in a clinic are infected with HIV. Furthermore, suppose that when a blood test for HIV is given, 98% of the patients infected with HIV test positive and that 3% of the patients not infected with HIV test positive. What is the probability that

- \( F \) has HIV
- \( F \) test positive
- \( \bar{F} \) doesn't have HIV
- \( \bar{F} \) test negative.

a. a patient testing positive for HIV with this test is infected with it?

\[ P(E) = .08 \quad P(F \mid E) = .98 \quad P(F \mid \bar{E}) = .03 \]

\[ P(E \mid F) = \frac{P(F \mid E)P(E)}{P(F \mid E)P(E) + P(F \mid \bar{E})P(\bar{E})} = \frac{.98(.08)}{(1-.02)(.08) + (.02)(.92)} \approx .74 \]

b. a patient testing negative for HIV with this test is infected with it?

\[ P(F \mid \bar{E}) = \frac{P(\bar{F} \mid E)P(E)}{P(\bar{F} \mid E)P(E) + P(\bar{F} \mid \bar{E})P(\bar{E})} = \frac{(.02)(.08)}{(.02)(.08) + (.97)(.92)} \approx .002 \]
7. What is the probability that the sum of the numbers on two dice is even when they are 
rolled? 6 \cdot 6 \text{ different ways} 
\text{Sum = even means both die are odd or both die are even} 
\begin{align*} 
3 \cdot 3 &= 9 \\
3 \cdot 3 &= 9 
\end{align*} 
\frac{18}{36} = \frac{1}{2} 

8. How many ways are there to assign three different jobs to five employees if each em-
ployee can be given no more than one job? 
\begin{align*} 
5 \cdot 4 \cdot 3 &= \boxed{60} 
\end{align*} 

9. A coin is biased so that the probability a head comes up when it is flipped is 0.6. What is 
the expected number of heads that come up when it is flipped 4 times? 
\begin{align*} 
X(s) &= \# \text{ of heads} \\
\begin{array}{c|c|c} 
X(s) & p(s) & X(s)p(s) \\
\hline 
0 & C(4,0)(.6)^0(.4)^4 & 0 \\
1 & C(4,1)(.6)^1(.4)^3 & C(4,1)(.6)(.4)^3 = 1.563 \\
2 & C(4,2)(.6)^2(.4)^2 & 2C(4,2)(.6)^2(.4)^2 = 0.912 \\
3 & C(4,3)(.6)^3(.4)^1 & 3C(4,3)(.6)^3(.4)^1 = 1.0368 \\
4 & C(4,4)(.6)^4(.4)^0 & 4C(4,4)(.6)^4(.4)^0 = 0.5184 \\
\hline 
\end{array} 
\end{align*} 
\boxed{2.4027} 

10. How many one-to-one functions are there from a set with five elements to a set with 
7 elements? 
\begin{align*} 
f: \{A, B, C, D, E\} \rightarrow \{x, y, z, w, v, u, t\} \\
7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 &= \boxed{2520} 
\end{align*}
11. What is the probability of the following events when we randomly select a permutation of \( \{1, 2, 3, 4\} \)?

a. 1 precedes 4.
\[ \frac{1}{2} \] 1/2 of them 1 comes before 4.

b. 4 precedes 1 and 4 precedes 2.
\[ \frac{8}{24} = \frac{1}{3} \] 
\[ \frac{3!}{3 \cdot 1 \cdot 2} \] when 4 is 1st + 3412 and 3421 8 total

c. 4 precedes 3 and 2 precedes 1.
\[ \frac{6}{24} = \frac{1}{4} \] 
4321, 4231, 4213 2143, 2413, 2431

12. Use the Binomial Theorem to show the following:
\[ \sum_{k=0}^{n} C(n, k) = 2^n. \]

**Binomial Theorem:** 
\[ (x+y)^n = \sum_{k=0}^{n} C(n, k) x^k y^{n-k} \]

Let \( x=1 \) and \( y=1 \).

Then \( (x+y)^n = 2^n \) and 
\[ \sum_{k=0}^{n} C(n, k) 1^k 1^{n-k} = \sum_{k=0}^{n} C(n, k) \]

Thus 
\[ \sum_{k=0}^{n} C(n, k) = 2^n \]

**BONUS QUESTION** (in the style of Math Jeopardy): This Swiss mathematician is best known for his posthumous work that described the known results in probability theory and in enumeration including the application of probability theory to games of chance and his introduction of the theorem known as the law of large numbers.

Who is Bernoulli?

(James or Jacob)