Now we consider the quantum mechanics of a single particle subject to a central force. The Hamiltonian operator is

$$\hat{H} = \hat{T} + \hat{V} = -(\hbar^2/2m) \nabla^2 + V(r) \quad (6.5)$$

where $\nabla^2$ is given by Eq. (3.55). Since $V$ is spherically symmetric, we will work in spherical polar coordinates. Hence, we want to transform the Laplacian operator to these coordinates. We already have the forms of the operators $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$ in these coordinates [Eqs. (5.91)–(5.93)], and by squaring each of these operators and then adding their squares, we get the Laplacian. This calculation is left as an exercise. The result is (Problem 6.12):

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \quad (6.6)$$

Looking back to (5.97), which gives the operator for the square of the magnitude of the orbital angular momentum of a single particle, $\hat{L}^2$, we see that

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} - \frac{1}{r^2 \hbar^2} \hat{L}^2 \quad (6.7)$$

The Hamiltonian (6.5) becomes

$$\hat{H} = \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \hat{L}^2 \right) + V(r) \quad (6.8)$$

In classical mechanics, a particle subject to a central force has its angular momentum conserved (Section 5.3). In quantum mechanics, we might ask whether we can have states with definite values for both the energy and the angular momentum. To have the set of eigenfunctions of $\hat{H}$ also be eigenfunctions of $\hat{L}^2$, the commutator $[\hat{H}, \hat{L}^2]$ must vanish. We have

$$[\hat{H}, \hat{L}^2] = [\hat{T}, \hat{L}^2] + [\hat{V}, \hat{L}^2] \quad (6.9)$$

$$[\hat{T}, \hat{L}^2] = \left[ -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) + \frac{1}{2mr^2} \hat{L}^2, \hat{L}^2 \right] \quad (6.10)$$

$$[\hat{T}, \hat{L}^2] = -\frac{\hbar^2}{2m} \left[ \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r}, \hat{L}^2 \right] + \frac{1}{2m} \left[ \frac{1}{r^2} \hat{L}^2, \hat{L}^2 \right] \quad (6.11)$$

Recall that $\hat{L}^2$ involves only $\theta$ and $\phi$ and not $r$ [Eq. (5.97)]. Hence, it commutes with any operator that involves only $r$. [To reach this conclusion we must use relations like (5.71) with $x$ and $z$ replaced by $r$ and $\theta$.] Thus the first commutator in (6.11) is zero. Moreover, since any operator commutes with itself, the second commutator in (6.11) is zero. Hence

$$[\hat{T}, \hat{L}^2] = 0 \quad (6.12)$$

Now, since $\hat{L}^2$ does not involve $r$, and $\hat{V}$ involves only $r$, we have

$$[\hat{V}(r), \hat{L}^2] = 0$$

Therefore

$$[\hat{H}, \hat{L}^2] = 0 \quad (6.13)$$

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